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Nonlinear stability of a weakly supercritical mixing layer in a rotating fluid

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A model has been constructed for a mixing layer of a rotating fluid with a large Reynolds number which is an analogue of a mixing-layer model for a plane flow widely used in the literature. The angular velocity profile in such a model has the form:

$$\Omega(r) = \tfrac{1}{2}(\Omega_1+\Omega_2)-\tfrac{1}{2}(\Omega_1-\Omega_2)\tanh\left(\frac{1}{D}\ln\frac{r}{R}\right),$$

where r is the distance from the rotation axis; and R, $\Omega_{1,2}$, and D are the model's parameters. The model permits a relatively simple analytical study of the stability for two-dimensional disturbances. It is shown that the stability is defined by the 'shear-width' parameter D, namely the model is unstable when $D < D_{\text{crit}} = \frac{1}{2}$. In a weakly supercritical flow ($|D-D_{\text{crit}}| \ll 1$), one mode with azimuthal number $m = 2$ develops. In this case two vortices are produced in the vicinity of a critical layer (CL), i.e. a radius where the wave's azimuthal velocity Ω_p coincides with the rotation velocity $\Omega(r)$. A study is made of their nonlinear evolution corresponding to different CL regimes: viscous, nonlinear, and unsteady. It is found that the instability saturates at a low enough level and the equilibrium amplitude depends on the degree of supercriticality $\Delta D = |D-D_{\text{crit}}|$, but the character of this dependence is different in different regions of the supercriticality parameter ΔD.

It is shown that, despite the specific form of the velocity profile in the model under consideration, results concerning the critical-layer dynamics have a high degree of universality. In particular, it becomes possible to formulate the criterion that the instability will be saturated at a low level for an arbitrary weakly supercritical flow.

1. Introduction

Models of plane-parallel mixing layers and, in particular, those with a velocity profile $u = \tanh y$ that have been widely used in the literature have one serious shortcoming as regards the possibility of providing a correct formulation of the problem of nonlinear stability within the framework of weakly nonlinear theory. The point here is that, for any plane mixing layer, there exists a critical wavelength, which in order of magnitude equals the thickness of the shear layer, such that modes with longer wavelengths are unstable. Thus, for the profile $u = \tanh y$ this critical wavelength is $\lambda_{\text{cr}} = 2\pi/k_{\text{cr}}$, where $k_{\text{cr}} = 1$. Every perturbation with $k < 1$ is unstable, and the largest growth rate approximately corresponds to the middle of the interval (0.1) and is not small, i.e. of order of magnitude unity (for more details see Michalke 1964). By invoking the method of the weakly nonlinear Stuart–Watson theory, a large number of authors have to resort to an artificial formulation of the problem and confine attention to the evolution of a mode with wavenumber k close to k_{cr} (Schade 1964; Huerre 1980, 1987; Benney & Maslowe 1975; Robinson 1974; Huerre & Scott 1980; Churilov & Shukhman 1987*c*). Also, one has to discard deliberately the fact that the system involves more unstable modes with smaller k, and this will inevitably distort the picture obtained.

It is clear that the above difficulties are avoided in the case where excitation of longer than critical wavelengths is prohibited. Such a situation naturally occurs with an axially symmetric mixing layer of a rotating fluid – there exist no perturbations with an azimuthal wavelength greater than $2\pi R$, where R is a certain typical radius on the flow profile. It is understandable that, if a weakly supercritical regime is possible in such a model, then it corresponds to excitation of a mode with the smallest m, i.e. $m = 1$. However, as more detailed analysis shows, the $m = 1$ mode is stable and, therefore, as it passes through the stability threshold, the $m = 2$ rather than $m = 1$ mode arises.

Thus, for the case of a rotating fluid the problem of nonlinear development of perturbations can be correctly formulated and solved in terms of a weakly nonlinear theory. We shall obtain evolution equations and shall follow the growth of perturbations from very small initial amplitudes up to stabilization of an instability.

Here, it seems appropriate to mention that, apart from that mentioned above, there is another factor that places constraints on the validity range of the evolution equations obtained in the papers cited above, for a plane flow with the profile $u = \tanh y$ to flows with an arbitrary form of the profile. The point here is that, within a mixing layer with the profile $u = \tanh y$, the critical layer that coincides with the inflection point $y = 0$ is also coincident with the symmetry point of the stream function. This leads to the fact that we cannot regard this case as a general one but rather, on the contrary, it is degenerate. Formally, this implies the possibility of introducing a real phase jump of the logarithm (as e.g. in Huerre & Scott 1980) or terms with the second time derivative for the regime with a nonlinear critical layer appearing in the evolution equation (as in the Benney & Maslowe's 1975 paper). These manifestations of symmetry of the profile $u = \tanh y$ are non-existent in the general case. With the model of an axially symmetric flow it becomes possible to avoid this degeneracy typical of a plane model with an antisymmetric velocity profile. A typical feature of the critical layer (CL) that does not coincide with the symmetry point, is its displacement along the direction of unperturbed velocity variation, in other words, there occurs not only a nonlinear growth of amplitude but also a nonlinear variation of the wave's phase velocity. Therefore, investigation of the weakly supercritical mixing layer of a rotating fluid is quite instructive in the study of nonlinear dynamics of critical layers because such a model combines all typical features of critical layers.

This paper is organized as follows. Section 2 gives results on linear theory of stability. In §3, we shall obtain the evolution equation for the amplitude and phase (or more specifically, for the phase velocity) for the viscous and nonlinear regimes of the critical layer. An analysis of these equations is made in §4. Section 5 investigates the regime of an unsteady CL as well as the transition from this regime to the regime of a nonlinear CL. Section 6 discusses the results obtained and provides an interpretation of the mechanism for nonlinear stabilization as well as offering a criterion for the low level of instability saturation for models with a sufficiently arbitrary velocity distribution.

2. Linear theory

We start from the equation for the stream function

$$\frac{\partial}{\partial t}\Delta\psi+\{\Delta\psi,\psi\} = \nu(\Delta^2\psi-\Delta^2\psi_{00}) \equiv \nu(\Delta^2\psi-\Delta\zeta_{00}), \tag{2.1}$$

where ψ_{00} corresponds to an unperturbed flow, and the last term on the right-hand side describes forces which give rise to the flow

$$\{a, b\} = \frac{1}{r}\frac{\partial a}{\partial \varphi}\frac{\partial b}{\partial r}-\frac{1}{r}\frac{\partial a}{\partial r}\frac{\partial b}{\partial \varphi}.$$

Assuming the Reynolds number to be very large, we drop the right-hand side of (2.1)† and linearize the equation. Assuming

$$\psi-\psi_{00} = \hat{\psi}(r)\,e^{im(\varphi-\Omega_p t)},$$

we get
$$r\frac{d}{dr}\left(r\frac{d\hat{\psi}}{dr}\right)-\left(m^2+\frac{r\dfrac{d}{dr}\zeta_{00}}{\Omega(r)-\Omega_p}\right)\hat{\psi} = 0. \tag{2.2}$$

Here
$$\Omega(r) = \frac{1}{r}\frac{d\psi_{00}}{dr}, \quad \zeta_{00} = \frac{1}{r}\frac{d}{dr}r^2\Omega$$

are the angular velocity and vorticity of the unperturbed flow. For the subsequent calculations, it is convenient to use the variable $y = \ln(r/R)$ instead of r:

$$\frac{d^2\hat{\psi}}{dy^2}-\left[m^2+\frac{\zeta'_{00}(y)}{\Omega(y)-\Omega_p}\right]\hat{\psi} = 0, \tag{2.3}$$

where $\zeta_{00} = \Omega'(y)+2\Omega(y)$, and the prime denotes a derivative with respect to y. Upon imposing the boundary conditions

$$\hat{\psi}\rightarrow 0 \quad \text{as} \quad y\rightarrow\pm\infty \tag{2.4}$$

we get a problem that is very close to the problem of a plane free mixing layer, with the exception that here the relation of $\zeta'_{00}(y)$ to $\Omega(y)$ is different from that holding for a plane flow, where $\zeta'_{00} = u''$. It is clear that, as in the plane case, neutral modes (and, therefore, the instability) are possible only in the presence of vorticity extrema, where $\zeta'_{00}(y) = 0$; however, these points now do not coincide with the inflection points. In the case of an arbitrary profile $\Omega(y)$ the solution of (2.3) is a task for a computer, but one may try to find a model that is solvable analytically. Therefore, it would be natural to attempt to use the following model:

$$\Omega(y) = \tfrac{1}{2}(\Omega_1+\Omega_2)-\tfrac{1}{2}(\Omega_1-\Omega_2)\tanh\left(\frac{y}{D}\right), \tag{2.5}$$

as such a reference model, in analogy with the plane case. It is easy to verify that, for such a model, the Sturm–Liouville problem for determining the neutral modes is, indeed, solved straightforwardly to give the following result:

† Of course, viscosity will be taken into account when solving the inner (i.e. within the CL) problem.

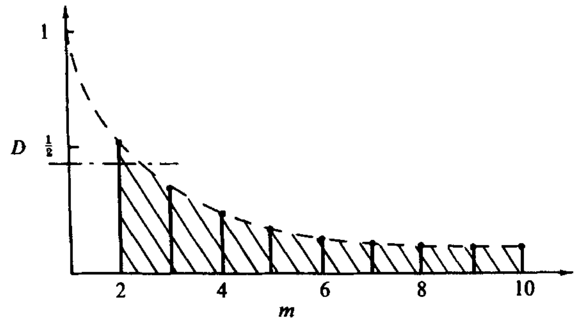

FIGURE 1. The regions of stability and instability (hatched) in linear theory.

1. phase velocity of neutral modes

$$\Omega_p = \Omega_n^{(m)}, \quad \Omega_n^{(m)} = \tfrac{1}{2}(\Omega_1 + \Omega_2) - \frac{1}{2m}(\Omega_1 - \Omega_2) \tag{2.6}$$

(index n denotes a neutral mode);

2. neutral curve

$$D = D_n \equiv \frac{1}{m}, \quad m = 1, 2, 3, \ldots; \tag{2.7}$$

3. eigenfunction of the neutral mode

$$\hat{\psi}(y) = \operatorname{sech}(my); \tag{2.8}$$

4. corotation radius y_{cn}, where $\Omega = \Omega_p$ (in other words, a critical level) is defined by the equality

$$\tanh(my_{cn}) = \frac{1}{m}. \tag{2.9}$$

The vorticity minimum lies at this point if the angular velocity is outwards ($\Delta\Omega = \Omega_1 - \Omega_2 > 0$), and a maximum is present if $\Delta\Omega < 0$. Note that y_{cn} does not coincide with the inflection point of the profile $\Omega(y), y = 0$, unlike the analogous plane model; however, as in the case of a plane model, y_{cn} is a regular point.

Thus, when the parameter of the shear width D† is equal to the value of $1/M$, then the mode with this value $m = M$ is a neutral one. From physical considerations it is quite clear that modes with smaller m, then, are unstable, while those with larger m are stable (see figure 1).

Now, let us calculate the increment and the correction to the phase velocity of the wave for a small deviation of D from its neutral value, i.e. when $D = D_n + \Delta D$. Assuming $\Omega_p = \Omega_n + \delta\Omega_p$, where $\delta\Omega_p = (\Delta\Omega_p)_L + (i/m)\gamma_L$ and using a standard perturbation procedure with the use of the Lin indentation rule, we obtain, for the increment γ_L and the correction to the wave's phase velocity $(\Delta\Omega_p)_L$,

$$\gamma_L = \frac{m^2|\Delta\Omega|\mu^2\Phi_1}{q(1+\mu^2\Phi_1\Phi_2)}\Delta D, \tag{2.10}$$

$$\frac{(\Delta\Omega_p)_L}{\Delta\Omega} = \left(\frac{m}{q}\frac{\mu}{1+\mu^2\Phi_1\Phi_2} - \frac{1}{2}\right)\Delta D. \tag{2.11}$$

† The physical shear width is related to the parameter D by the relationship $\Delta R \sim R\sinh(D)$.

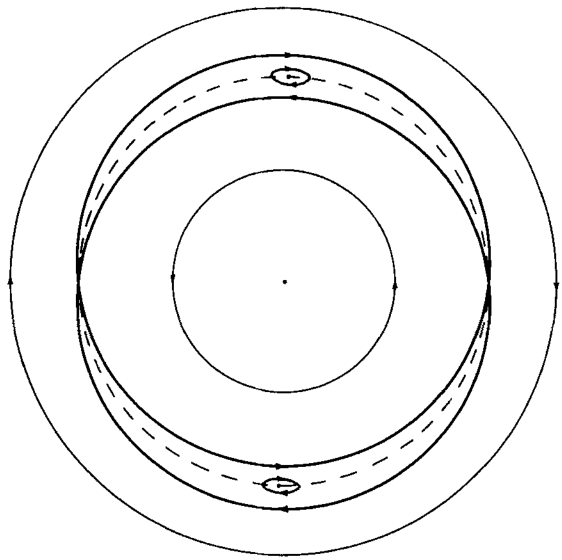

FIGURE 2. Streamlines for the neutral mode $m = 2$ in a frame reference rotating with phase velocity Ω_p.

Here $$\mu = q\Big/\left(\frac{2}{m}+q\lambda_m\right), \quad \lambda_m = \ln\frac{m+1}{m-1}, \quad q = 1-\frac{1}{m^2}, \quad \Delta\Omega = \Omega_1-\Omega_2,$$

and Φ_1 and Φ_2 represent phase jumps of the logarithms arising when the indentation rule is being employed. As is known, linear theory gives $\Phi_1 = \Phi_2 = -\pi$.†

From (2.10) it is evident that the increment is positive as $\Delta D < 0$, as one might expect. From (2.11) it follows that for the case of a decreasing angular velocity ($\Delta\Omega > 0$) the phase velocity for the unstable mode is larger than the neutral-mode phase velocity, and vice versa.

Let us examine the $m = 1$ mode. For it, $\gamma_L = 0$, i.e. it remains neutral even with a decrease of D from the value of $D_n^{(1)} = 1$. More detailed analysis of this mode reveals that this is a neutral mode for any D. The relevant eigenfunction has the form

$$\hat{\psi}_{m-1} = \operatorname{sech}\left(\frac{y}{D}\right)\mathrm{e}^{-y(1-1/D)}, \tag{2.12}$$

and for any $D < 2$, it satisfies the boundary conditions. Thus, a critical value that separates stable flows from unstable flows, is the value of $D_{crit} = \frac{1}{2}$ (rather than $D_{crit} = 1$, as one might anticipate from the neutral curve only).

In the weakly supercritical case of interest in the subsequent discussion the $m = 2$ mode arises. In this case near the corotation radius ($r = 3^{\frac{1}{4}}R$), a system of two vortices appear, anticyclonic if $\Delta\Omega > 0$ and cyclonic if $\Delta\Omega < 0$ (figure 2).

It is useful to remember that in an unstable flow the critical level no longer coincides with the position of the vorticity extremum but is separated from it by the interval $\Delta y = (\Delta D + 2(\Delta\Omega_p)_L/\Delta\Omega)\,q^{-1}$ so that at the critical level $\zeta'_{00} \neq 0$ (see figure 3), and the instability increment is proportional to the value of ζ'_{00} at the critical level.

† We have introduced deliberately two different designations for phase jumps of the sine and cosine part of the wave because in nonlinear theory they will represent two different functions of amplitude (see the next Section).

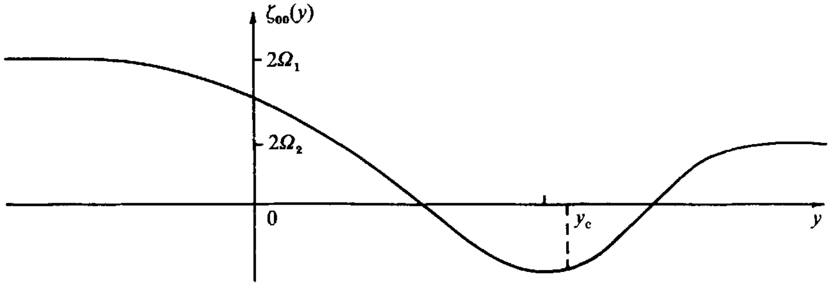

FIGURE 3. Schematic representation of the unperturbed vorticity profile. (The y_c-coordinate corresponds to the initial position of the critical level.)

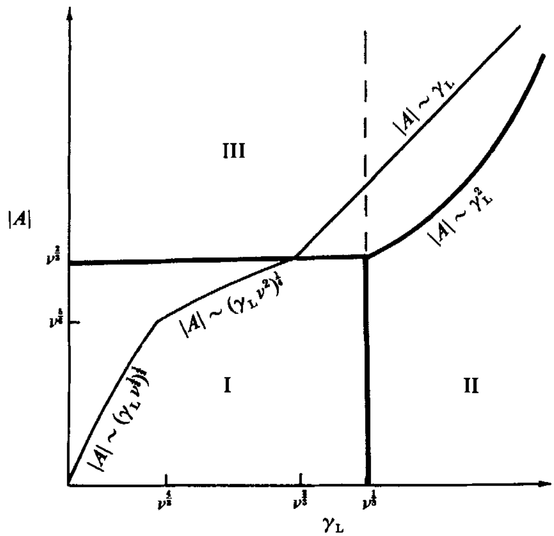

FIGURE 4. Amplitude–supercriticality diagram: I, region of a viscous CL; II, region of an unsteady CL; III, region of a nonlinear CL. The boundaries of regions I, II and III are indicated by a heavy line. The saturation amplitude as a function of supercriticality is shown.

3. Evolution equations for the regimes of a viscous and nonlinear CL

A dominating factor that determines the course of the nonlinear evolution is the rearrangement of the critical layer. For an equation of the form (2.3) (i.e. with a singularity of the first-order-pole type) a classification of critical layers is shown on the amplitude–supercriticality diagram (figure 4). According as which of the three scales is greater: $l_\nu = \nu^{\frac{1}{3}}$, $l_t = \gamma$, or $l_N = A^{\frac{1}{2}}$, the critical layer can be a viscous, unsteady or nonlinear one, respectively.

We wish to follow the evolution of an initially small perturbation, whose amplitude in the initial stage corresponds to the lower part of the figure 4 diagram. A distinction should then be drawn between two cases. In the first case supercriticality corresponds to the region of a viscous CL, i.e. $\gamma_L < \nu^{\frac{1}{3}}$, and in the second case it corresponds to the region of an unsteady CL, i.e. $\gamma_L > \nu^{\frac{1}{3}}$. (Here we are already using dimensionless quantities, by measuring time in $|\Delta\Omega|^{-1}$ units and viscosity in $|\Delta\Omega| R^2$ units.) The character of the perturbation growth is different in each case. In this Section we shall study the evolution of the perturbations that start from region I, i.e.

from the region of a viscous CL. However, we shall see that, even when starting from region II (of an unsteady CL), the perturbation, as it evolves, reaches region III (of a nonlinear CL) so that the equations to be derived in this Section are applicable to region I and to the whole of region III of the diagram, and not only to the part lying above the region of a viscous CL; but it does not involve the description of the transition from II to III. The evolution of a perturbation starting from region II will be considered in §5.

Now, we begin to derive the evolution equations. We put

$$D = \tfrac{1}{2}+\epsilon D_1, \quad \frac{\partial}{\partial t} = \epsilon\frac{\partial}{\partial \tau}-\Omega_n\frac{\partial}{\partial \varphi}, \quad \nu = \epsilon^{\frac{3}{2}}\eta,$$

where ϵ is a small parameter characterizing the perturbation amplitude, $D_1 < 0$. The derivation procedure has been described in several papers. A detailed description may be found in, for example, Churilov & Shukhman (1987*a*). We shall give a brief description. The solution of (2.1) is sought in the form

$$\left.\begin{aligned} \psi &= \Psi+\psi_{00}+\tfrac{1}{2}\sigma\int \mathrm{d}y\,\mathrm{e}^{2y}\left(\tanh(my)-\frac{1}{m}\right), \\ \Psi &= \sum_{l=-\infty}^{\infty} \psi_l(\tau, y)\,\mathrm{e}^{\mathrm{i}ml(\varphi-\Omega_n t)}, \end{aligned}\right\} \tag{3.1}$$

where $\psi_l = \bar{\psi}_{-l}$, the overbar denotes complex conjugation, $m = 2$ (in the case of interest), $\sigma = \operatorname{sgn}(\Delta\Omega)$, and

$$\psi_{00} = \int \mathrm{e}^{2y}\Omega(y)\,\mathrm{d}y = \int \mathrm{e}^{2y}\left[\Omega_n+\frac{\sigma}{2}\left(\frac{1}{m}-\tanh\left(\frac{y}{D}\right)\right)\right]\mathrm{d}y.$$

The functions ψ_l are sought in the form of an expansion into a power series:

$$\begin{aligned} \psi_1 &= \epsilon\psi_1^{(1)}+\epsilon^2\psi_1^{(2)}+\dots, \\ \psi_2 &= \epsilon^2\psi_2^{(2)}+\dots, \\ \psi_0 &= \epsilon^2\psi_0^{(2)}+\dots. \end{aligned}$$

At $O(\epsilon)$, for the fundamental harmonic we obtain

$$\psi_1^{(1)} = A(\tau)\operatorname{sech}(my) \equiv A(\tau)\varphi_a(y),$$

where $A(\tau)$ is a complex amplitude of the wave which is convenient to represent by separating in explicit form the modulus and the argument:

$$A(\tau) = q^{-\frac{1}{2}}C(\tau)\,\mathrm{e}^{-\mathrm{i}m\chi(\tau)}. \tag{3.2}$$

The evolution equation will be obtained for the real amplitude $C(\tau)$ and phase $\theta = m(\varphi-\chi(\tau)-\Omega_n t)$ or more exactly, for the addition to the wave's phase velocity

$$\omega(\tau) = \frac{\mathrm{d}\chi}{\mathrm{d}\tau}.$$

At $O(\epsilon^2)$ of the fundamental we obtain

$$\left\{\frac{\partial^2}{\partial z^2}-\left(1-\frac{2}{\cosh^2 z}\right)\right\}\psi_1^{(2)} = Q_1^{(2)}(z), \tag{3.3}$$

where

$$Q_1^{(2)}(z) = \left(\frac{8\mathrm{i}}{m\sigma}\frac{\partial}{\partial\tau}+D_1\right)\frac{2\psi_1^{(1)}}{\cosh^2 z}\left(\tanh z-\frac{1}{m}\right)^{-1}+4mD_1\frac{1-z\tanh z}{\cosh^2 z}\psi_1^{(1)}, \quad z = my.$$

Owing to singularity at point z_c, where $\tanh z_c = m^{-1}$ (3.3) must be solved separately to the right and to the left of the CL, and then the solutions must be matched, by solving preliminarily the problem inside the CL. For $\psi_1^{(2)}$ we get

$$\psi_1^{(2)}(z) = a^{\pm}A\varphi_a(z)+\varphi_a(z)\int_{z_c}^{z}(F_1^{(2)}(s)-b^{\pm})\varphi_a^{-2}(s)\,\mathrm{d}s, \tag{3.4}$$

where $a^{\pm}(\tau)$ and $b^{\pm}(\tau)$ are unknown coefficients derivable from the inner problem, and

$$F_1^{(2)}(z) = \int^{z}\varphi_a(s)\,Q_1^{(2)}(s)\,\mathrm{d}s.$$

From this, bearing in mind that $\psi_1^{(2)}\to 0$ as $z\to\pm\infty$, we obtain the so-called modified solvability condition (MSC):

$$F_1^{(2)}(\infty)-F_1^{(2)}(-\infty) = (b^+-b^-)A,$$

or in explicit form

$$(b^+-b^-)A = 4mD_1A-\frac{2q}{\mu}\left(D_1A+\frac{2\mathrm{i}}{m\sigma}\frac{\mathrm{d}A}{\mathrm{d}\tau}\right). \tag{3.5}$$

From the outer solution, apart from the MSC (3.5), we also require the inner asymptotic expansion, i.e. one for $z\to z_c$. Assuming $z-z_c = \epsilon Y$, we write the final form of this asymptotic expansion in explicitly real form:

$$\Psi = \epsilon(\Psi^{(0)}+\epsilon^{\frac{1}{2}}\Psi^{(\frac{1}{2})}+\epsilon\Psi^{(1)}+\epsilon^{\frac{3}{2}}\Psi^{(\frac{3}{2})}+\dots), \tag{3.6}$$

where

$$\left.\begin{aligned}
\Psi^{(0)} &= -\frac{\sigma}{4m}q\rho Y^2+2C\cos\theta,\\
\Psi^{(\frac{1}{2})} &= -\frac{\sigma}{6m^2}q\rho Y^3-\frac{2}{m}CY\cos\theta,\\
\Psi^{(1)} &= \frac{\sigma}{24m}\left(1-\frac{3}{m^2}\right)q\rho Y^4-\left(1-\frac{2}{m^2}\right)CY^2\cos\theta+a_2^{(2)}C^2\cos 2\theta,\\
\Psi^{(\frac{3}{2})} &= -\frac{\sigma}{30m^4}q\rho Y^5+\frac{1}{3m}\left(5-\frac{6}{m^2}\right)CY^3\cos\theta-\frac{16}{\sigma\rho}C^2YA(Y)\cos 2\theta\\
&\quad+Y\left\{\left[2\left(D_1C+\frac{2\mathrm{i}}{m\sigma}\frac{\mathrm{d}C}{\mathrm{d}\tau}\right)(A(Y)-1)-b^{\pm}Cq\right]\mathrm{e}^{\mathrm{i}\theta}+\text{c.c.}\right\}.
\end{aligned}\right\} \tag{3.7}$$

Here $\rho = \exp(2z_c/m) = \dfrac{m+1}{m-1}$, $A(Y) = \ln(q\epsilon^{\frac{1}{2}}|Y|)$.

Let us now examine the inner region. We put

$$\Psi = \epsilon\tilde{\Psi} = \epsilon(\tilde{\Psi}^{(0)}+\epsilon^{\frac{1}{2}}\tilde{\Psi}^{(\frac{1}{2})}+\epsilon\tilde{\Psi}^{(1)}+\epsilon^{\frac{3}{2}}\tilde{\Psi}^{(\frac{3}{2})}+\dots).$$

For $\tilde{\Psi}$, we obtain the equation†

$$\begin{aligned}
&\tilde{\Psi}'\tilde{\Psi}''_{\theta}-\tilde{\Psi}'''\tilde{\Psi}_{\theta}+\epsilon(\tilde{\Psi}'_{\theta}\tilde{\Psi}_{\theta\theta\theta}-\Psi'_{\theta\theta}\Psi_{\theta})+\frac{2}{m}\epsilon^{\frac{3}{2}}\tilde{\Psi}_{\theta\theta}\tilde{\Psi}_{\theta}+\epsilon^{\frac{1}{2}}qz_{c}\,m\rho D_{1}\tilde{\Psi}''_{\theta}\\
&\quad+\epsilon^{\frac{1}{2}}\frac{\rho}{m^{2}}\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{\Psi}''+\frac{2}{m}\epsilon^{\frac{3}{2}}\tilde{\Psi}_{\theta\theta}\tilde{\Psi}_{\theta}+\tfrac{1}{2}\epsilon^{\frac{3}{2}}\sigma z_{c}\,q\rho D_{1}\tilde{\Psi}_{\theta\theta\theta}+\epsilon^{\frac{3}{2}}\sigma\left(\frac{2q}{m}+q^{2}z_{c}\right)D_{1}\tilde{\Psi}_{\theta}\\
&\quad+\epsilon^{\frac{3}{2}}\frac{\rho}{m^{2}}\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{\Psi}_{\theta\theta}\\
&=\eta\left\{\tilde{\Psi}''''-\frac{4}{m}\epsilon^{\frac{1}{2}}\tilde{\Psi}'''+2\epsilon\tilde{\Psi}''_{\theta\theta}+\frac{4}{m^{2}}\epsilon\tilde{\Psi}''-\frac{4}{m}\epsilon^{\frac{3}{2}}\tilde{\Psi}'_{\theta\theta}\right.\\
&\quad\left.-\rho^{2}\left[(\Delta_{y}\zeta_{00})_{c}\,\epsilon+\epsilon^{\frac{3}{2}}Y\left(\frac{4}{m}(\Delta_{y}\zeta_{00})_{c}+(\Delta_{y}\zeta_{00})'_{c}\right)\right]\right\}.
\end{aligned}\qquad(3.8)$$

Here $\Delta_y = \mathrm{e}^{-2y}\,\mathrm{d}^2/\mathrm{d}y^2$ is the radial part of the Laplace operator,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}=\frac{\partial}{\partial\tau}-m\frac{\mathrm{d}\chi}{\mathrm{d}\tau}\frac{\partial}{\partial\theta}=\frac{\partial}{\partial\tau}-m\omega(\tau)\frac{\partial}{\partial\theta},$$

and the prime indicates the derivative with respect to Y.

In the model considered we have

$$(\Delta_y\zeta_{00})_c=\rho^{-1}\sigma q^2m^3,\quad (\Delta_y\zeta_{00})'_c=-8\rho^{-1}\sigma q^2m^2.$$

From (3.8) we get $\tilde{\Psi}^{(0)}=\Psi^{(0)}$, $\tilde{\Psi}^{(\frac{1}{2})}=\Psi^{(\frac{1}{2})}$, and $\tilde{\Psi}^{(1)}=\Psi^{(1)}$, and for $\tilde{\Psi}^{(\frac{3}{2})}$ we obtain the equation

$$\begin{aligned}
&-\frac{\sigma}{2m}q\rho Y\frac{\partial\tilde{\zeta}}{\partial\theta}+2C\sin\theta\frac{\partial\tilde{\zeta}}{\partial Y}-\eta\frac{\partial^2\zeta}{\partial Y^2}\\
&\qquad=-\frac{16}{m}qC^2\sin 2\theta+\frac{4q}{m^2}\rho\left[\cos\theta\frac{\mathrm{d}C}{\mathrm{d}\tau}+mC(\omega+\tfrac{1}{2}D_1\sigma)\sin\theta\right],
\end{aligned}\qquad(3.9)$$

where

$$\tilde{\zeta}=\tilde{\Psi}^{(\frac{3}{2})''}+\frac{2}{3}\frac{\sigma}{m^4}q\rho Y^3-\frac{2}{m}\left(5-\frac{6}{m^2}\right)CY\cos\theta.\qquad(3.10)$$

Equation (3.9) must be solved with the asymptotic behaviour

$$\tilde{\zeta}\sim-\frac{16}{\rho\sigma}C^2Y^{-1}\cos 2\theta+Y^{-1}\left\{4\left(D_1+\frac{2\omega}{\sigma}\right)C\cos\theta-\frac{8}{m\sigma}\frac{\mathrm{d}C}{\mathrm{d}\tau}\sin\theta\right\}.\qquad(3.11)$$

With the aid of (3.7) and (3.10) we rewrite the MSC (3.5) as

$$\int_{-\infty}^{\infty}\langle\tilde{\zeta}\cos\theta\rangle\,\mathrm{d}Y=-\frac{4m}{q}D_1C+\frac{4}{\mu}\left(\tfrac{1}{2}D_1+\frac{\omega}{\sigma}\right)C,\qquad(3.12)$$

$$\int\langle\tilde{\zeta}\sin\theta\rangle\,\mathrm{d}Y=-\frac{4}{\mu m\sigma}\frac{\mathrm{d}C}{\mathrm{d}\tau}.\qquad(3.13)$$

† Note the presence of the third derivative of unperturbed vorticity on the right-hand side of (3.8), with which – as will be shown later in the text – the stabilizing effect of nonlinearity, in the presence of viscosity, will be associated.

Here $$\int_{-\infty}^{\infty} \mathrm{d}Y(\ldots) = \lim_{T\to\infty} \int_{-T}^{T} \mathrm{d}Y(\ldots), \quad \langle\ldots\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta(\ldots).$$

Thus, (3.9), (3.12) and (3.13) are a starting system of equations for obtaining the evolution equations, i.e. the equation for the amplitude $C(\tau)$ and the addition to the phase velocity $\omega(\tau)$. On solving (3.9) with the asymptotic behaviour (3.11) and substituting into (3.12) and (3.13), we obtain the desired evolution equations.

The procedure of determining $\tilde{\zeta}$ from (3.9) and evaluating the integrals $\int\langle\tilde{\zeta}\sin\theta\tilde{\zeta}\rangle\,\mathrm{d}Y$ and $\int\langle\tilde{\zeta}\cos\theta\rangle\,\mathrm{d}Y$ involved in the MSC, is deferred to the Appendix. We give the outcome:

$$\int\langle\tilde{\zeta}\cos\theta\rangle\,\mathrm{d}Y = -4C\frac{\gamma(\tau)}{m}\Phi_2\left(\frac{C}{(p\eta^2)^{\frac{1}{3}}}\right), \tag{3.14}$$

$$\int\langle\tilde{\zeta}\sin\theta\rangle\,\mathrm{d}Y = -4C\sigma\left(\frac{D_1}{2}+\frac{\omega}{\sigma}\right)\Phi_1\left(\frac{C}{(p\eta^2)^{\frac{1}{3}}}\right)+\frac{8C^2}{\sigma\rho}\Phi_3\left(\frac{C}{(p\eta^2)^{\frac{1}{3}}}\right). \tag{3.15}$$

It should be noted that the contribution involving Φ_3 is due to taking account of cubic expansion terms of unperturbed vorticity near the CL or, more exactly, of linear expansion terms from the Laplacian of unperturbed vorticity

$$\sim(\Delta_y\zeta_{00})'_{\mathrm{c}}(y-y_{\mathrm{c}}).$$

One may trace from the beginning† that the coefficient of $C^2\Phi_3$ in (3.15) is, in fact, proportional to the value of $(\Delta_y\zeta_{00})'_{\mathrm{c}}$, i.e.

$$\lambda = \frac{8}{\sigma\rho} = \frac{-1}{q^2m^3}\left[\frac{\mathrm{d}}{\mathrm{d}y}(\Delta_y\zeta_{00})\right]_{\mathrm{c}}. \tag{3.16}$$

In (3.14) and (3.15) $\gamma(\tau) = (\mathrm{d}C/\mathrm{d}\tau)C^{-1}$, and $p = q\rho/2m$, while $\Phi_1(x)$, $\Phi_2(x)$, and $\Phi_3(x)$ are certain universal functions, the first of which is related to the known function of phase jump as obtained by Haberman (1972), i.e. $\Phi_1(x) = \Phi_{\mathrm{H}}(x^{-\frac{3}{2}})$, where $\Phi_{\mathrm{H}}(\lambda_{\mathrm{c}})$ is the Haberman function, and $\Phi_2(x)$ and $\Phi_3(x)$ are two new functions which arise owing to the difference of equation (3.9) from a corresponding equation of Haberman's paper in that the right-hand side of (3.9) involves terms with $\cos\theta$ and $\sin 2\theta$, respectively. (The function $\Phi_3(x)$ has also been obtained in a recent paper by Churilov 1988.) We give the asymptotics of these functions corresponding to regions of a viscous CL ($C/\eta^{\frac{2}{3}} \ll 1$, i.e. $x \ll 1$) and a nonlinear CL ($C/\eta^{\frac{2}{3}} \gg 1$, i.e. $x \gg 1$):

$$\Phi_1(x) = \begin{cases} -\pi+I_0x^2 & \text{at} \quad x \ll 1, \quad I_0 = 6.42 \\ -I_1x^{-\frac{3}{2}} & \text{at} \quad x \gg 1, \quad I_1 = 0.621\pi,‡ \end{cases} \tag{3.17}$$

† Note that in this case the first three terms involved in the formula for $\Psi^{(\frac{3}{2})}$ in (3.7) which are, respectively, the expansion terms of the unperturbed stream function, the eigenfunction of the neutral mode and the eigenfunction of the second harmonic, will be written as

$$\rho\left[\frac{\rho}{m^4}(\Delta_y\zeta_{00})'_{\mathrm{c}}+\frac{4\sigma}{m^2}q\left(2-\frac{3}{m^2}\right)\right]\frac{Y^5}{5!}-\frac{1}{3m}\left[3-\frac{2}{m^2}+\frac{\sigma\rho}{qm^2}(\Delta_y\zeta_{00})'_{\mathrm{c}}\right]CY\cos\theta+\frac{2}{q^2m^2}(\Delta_y\zeta_{00})'_{\mathrm{c}}C^2YA(Y)\cos 2\theta,$$

while the term involving $\sin 2\theta$ on the right-hand side of (3.9) will be written as

$$\frac{2\rho}{qm^3\sigma}(\Delta_y\zeta_{00})'_{\mathrm{c}}\sin 2\theta.$$

‡ Haberman (1972) gives for I_1 a correct analytical expression in the form of an integral; however, this integral was not correctly evaluated numerically.

$$\Phi_2(x) = \begin{cases} -\pi - I_0 x^2 & \text{at} \quad x \ll 1, \\ -I_2 x^{\frac{3}{2}} & \text{at} \quad x \gg 1, \quad I_2 = 14.10, \end{cases} \tag{3.18}$$

$$\Phi_3(x) = \begin{cases} I_4 x^3 & \text{at} \quad x \ll 1, \quad I_4 = 1.12\pi \\ I_3 x^{-\frac{3}{2}} & \text{at} \quad x \gg 1. \quad I_3 = 4.26. \end{cases} \tag{3.19}$$

Substitution of (3.14) and (3.15) into (3.12) and (3.13) gives the evolution equations (which we write in physical variables

$$t = \tau/\epsilon, \quad B = \epsilon C, \quad \Delta\Omega_{\mathrm{p}} = \epsilon\omega, \quad \text{and} \quad \Delta D = \epsilon D_1):$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \gamma_{\mathrm{N}}(t)\, B(t), \tag{3.20}$$

$$\Delta\Omega_{\mathrm{p}}(t) = \sigma\left\{\left(\frac{1}{2} - \frac{m\mu}{q}\right)|\Delta D| - \frac{\mu}{m}\gamma_{\mathrm{N}}(t)\,\Phi_2(x)\right\}, \tag{3.21}$$

and the nonlinear increment $\gamma_{\mathrm{N}}(t)$ is

$$\gamma_{\mathrm{N}}(t) = m\frac{(m\mu^2/q)\,\Phi_1(x)\,\Delta D - (\lambda\mu\sigma/4)\,\Phi_3(x)\,B}{1 + \mu^2\Phi_1(x)\,\Phi_2(x)}, \tag{3.22}$$

where $x = B(t)/(\nu^2 p)^{\frac{1}{3}}$. (It should be borne in mind that q, ρ, μ, and p are constants which in the case of the mode $m = 2$ under consideration are $q = \frac{3}{4}$; $\rho = 3$; $\mu = 0.4112$; and $p = \frac{9}{16}$).

The expression (3.22) for the increment requires a comment. If for the moment the contribution involving Φ_3 is neglected, then we shall see that the difference of the nonlinear increment from the linear one (2.10) lies in the replacement of phase jumps $\Phi_1 = -\pi$ and $\Phi_2 = -\pi$ by their nonlinear values. Such a nonlinear reduction of Φ_1 and Φ_2 decreases the increment but does not make it zero. The situation is altered if the contribution involving Φ_3 is taken into account. It has a negative sign in the present case and gives the stability. We again remind the reader that the origin of this contribution is attributable to taking account of cubic terms of the expansion of unperturbed vorticity near the singularity, and which becomes important as the amplitude grows and the CL increases in width. In §6 we shall attempt to explain the physical sense of this stabilization.

Next, we examine the evolution of the perturbations in greater detail.

4. The evolution of perturbations in regimes of a viscous and nonlinear CL

Equations (3.20)–(3.22) derived above permit us to study the dynamics of perturbations from the initial small level to stabilization of the instability.

4.1. *The regime of a viscous CL* ($B \ll \nu^{\frac{2}{3}}$, $\gamma_{\mathrm{L}} \ll \nu^{\frac{1}{3}}$)

For the regime of a viscous CL, with the help of (3.17)–(3.19) we get

$$\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t} = \gamma_{\mathrm{L}}\left(1 - I_0\frac{B^2}{(p\nu^2)^{\frac{1}{3}}}\right) - \frac{4I_4}{\rho^2 q}\frac{\mu m^2}{1+\pi^2\mu^2}\frac{B^4}{\nu^2}, \tag{4.1}$$

$$\Delta\Omega_{p}(t) = (\Delta\Omega_{p})_{L} - \frac{4\pi m\mu^{2}\sigma I_{4}}{q\rho^{2}(1+\pi^{2}\mu^{2})}\frac{B^{4}}{\nu^{2}}, \tag{4.2}$$

where

$$\gamma_{L} = \frac{m^{2}\mu^{2}\pi|\Delta D|}{q(1+\pi^{2}\mu^{2})} = 1.06|\Delta D|, \quad (\Delta\Omega_{p})_{L} = \sigma\left(\frac{1}{2} - \frac{m}{q}\frac{\mu}{1+\pi^{2}\mu^{2}}\right)|\Delta D| \approx 0.09\sigma|\Delta D|$$

are the increment and the addition to phase velocity in linear theory. In the regime of a viscous CL, the cubic nonlinearity involved in (4.1) is not competitive (i.e. the cubic nonlinear term is much smaller than the linear one) and nonlinearity B^{5} is a dominant nonlinear effect. It has a stabilizing sign and bounds the amplitude growth at the level

$$B_{\text{sat}}^{(1)} = 1.15(\gamma_{L}\nu^{2})^{\frac{1}{4}}. \tag{4.3}$$

At the saturation stage the phase velocity is

$$\Omega_{p} = (\Omega_{p})_{\text{sat}} = \Omega_{n} + (\Delta\Omega_{p})_{\text{sat}}, \quad (\Delta\Omega_{p})_{\text{sat}} = \sigma\left(\frac{1}{2} - \frac{m\mu}{q}\right)|\Delta D| = -0.60\sigma|\Delta D|. \tag{4.4}$$

From (4.3) it follows that stabilization does, indeed, occur in the regime of a viscous CL, i.e. $B_{\text{sat}}^{(1)}/\nu^{\frac{2}{3}} \ll 1$, only for supercriticalities $\gamma_{L} < \nu^{\frac{2}{3}}$. When $\gamma_{L} > \nu^{\frac{2}{3}}$, nonlinearity becomes insufficiently strong so that the perturbation goes over into the regime of a nonlinear CL:

4.2. *The regime of a nonlinear CL* ($B \gg \nu^{\frac{2}{3}}$)

In this region, the evolution equations obtainable with the aid of (3.17)–(3.19) have the form

$$\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\nu}{B^{\frac{3}{2}}}\frac{\mu}{(q\rho/2m)^{\frac{1}{2}}}\frac{\frac{1}{2}I_{1}\mu m\rho|\Delta D| - qI_{3}B(t)}{1+\mu^{2}I_{1}I_{2}}, \tag{4.5}$$

$$\Delta\Omega_{p} = (\Delta\Omega_{p})_{\text{sat}} + 2\frac{\sigma\mu^{2}I_{2}}{q\rho}\frac{\frac{1}{2}I_{1}\mu m\rho|\Delta D| - qI_{3}B(t)}{1+\mu^{2}I_{1}I_{2}}. \tag{4.6}$$

The evolution of perturbations in the case $\gamma_{L} > \nu^{\frac{2}{3}}$ (but $\gamma_{L} < \nu^{\frac{1}{3}}$) looks like this. The amplitude actually grows exponentially with growth rate γ_{L} up to the boundary with the nonlinear CL, i.e. $B \sim \nu^{\frac{2}{3}}$ because here nonlinearity still is not competitive. As the amplitude reaches values corresponding to the nonlinear CL, but still is far from the saturation amplitude $B = B_{\text{sat}}^{(2)}$,

$$B_{\text{sat}}^{(2)} = \frac{1}{2}\frac{\mu m\rho}{q}\frac{I_{1}}{I_{3}}|\Delta D| = 1.14\gamma_{L}, \tag{4.7}$$

the growth rate decreases: $\gamma \to \gamma_{L}(\nu/B^{\frac{3}{2}})$. In this case the variation of amplitude becomes a power-law: $B \sim t^{\frac{2}{3}}$ and the correction to the phase velocity assumes the value

$$\Delta\Omega_{p} = (\Delta\Omega_{p})^{*} = \sigma\left(\frac{1}{2} - \frac{\mu m}{q(1+\mu^{2}I_{1}I_{2})}\right)|\Delta D| = 0.31|\Delta D|. \tag{4.8}$$

Finally, when the amplitude reaches values of $B \sim \gamma_{L}$, the growth becomes slower and the amplitude arrives at its final value $B_{\text{sat}}^{(2)}$. The phase velocity also assumes its final value $(\Omega_{p})_{\text{sat}}$, (4.4).

It is interesting to follow the displacement of the critical level whose position is

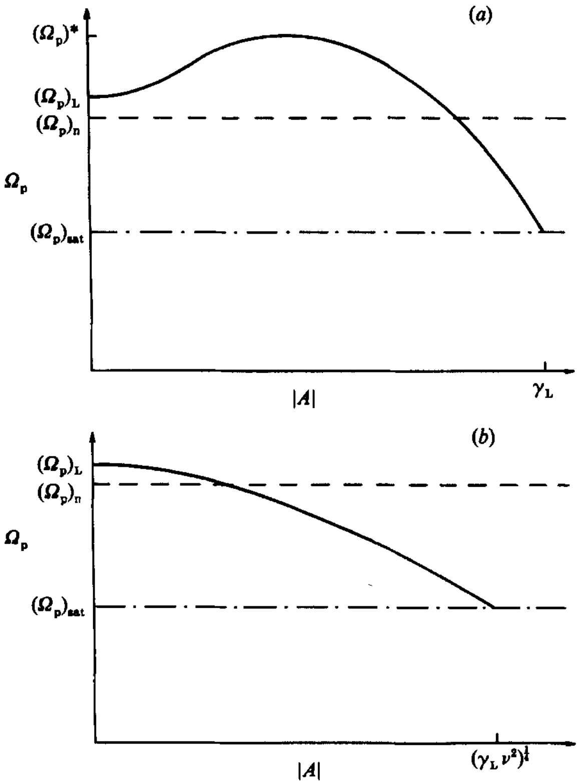

FIGURE 5. Phase velocity Ω_p as a function of amplitude ($\sigma > 0$): (*a*) $\gamma_L > \nu^{\frac{2}{3}}$; (*b*) $\gamma_L < \nu^{\frac{2}{3}}$.

related to the phase velocity by the relationship $\Omega_p = \Omega(y)$. During the course of its evolution the critical level is displaced along y the distance

$$y_{sat} - y_L = -\frac{2\sigma}{q}[(\Delta\Omega_p)_{sat} - (\Delta\Omega_p)_L] = \frac{2m\mu^3}{q^2}\frac{|\Delta D|}{1+\pi^2\mu^2}, \tag{4.9}$$

where y_{sat} and y_L are the final and the initial positions of the critical level, respectively. The critical level is displaced to the side of the rotation axis, irrespective of the sign of the angular velocity gradient σ. The phase velocity decreases as compared with the linear value for the case of a falling angular velocity ($\sigma > 0$) and increases if the angular velocity grows outwards ($\sigma < 0$). The dependence of the phase velocity on amplitude is shown in figure 5.

4.3. *The intermediate region* ($B \sim \nu^{\frac{2}{3}}$)

If supercriticality assumes intermediate values, i.e. $\gamma_L \sim \nu^{\frac{2}{3}}$, the saturation amplitude finds itself at the boundary of the viscous and nonlinear CL regions. It is defined by the relation $\gamma_N = 0$ which is convenient to write as

$$-\frac{x\Phi_3(x)}{\Phi_1(x)} = 1.88\left(\frac{\gamma_L}{\nu^{\frac{2}{3}}}\right), \tag{4.10}$$

where $x = B_{sat}/(p\nu^2)^{\frac{1}{3}}$. In order to determine numerically B_{sat} in this intermediate region, it is necessary to know the functions $\Phi_1(x)$ and $\Phi_3(x)$ for values of x of the order unity. For the function $\Phi_3(x)$, we have no such information available, although, in principle, $\Phi_3(x)$ can be calculated for all x. However, the picture is quite clear qualitatively without these calculations as well. In figure 4, the saturation amplitude as a function of supercriticality γ_L is plotted for all values of γ_L, including those in the intermediate region $\gamma_L \sim \nu^{\frac{2}{3}}$, where $B_{sat} \sim \nu^{\frac{2}{3}}$.

To conclude this Section, we wish to note that, strictly speaking, the results obtained here refer only to the part of the figure 4 diagram for which $\gamma_L < \nu^{\frac{1}{3}}$, i.e. to the left of the dashed line. However, we may also continue the saturation amplitude curve $B_{sat} \sim \gamma_L$ into the region $\gamma_L > \nu^{\frac{1}{3}}$ because, as will be shown below, from the region of an unsteady CL ($\gamma_L > \nu^{\frac{1}{3}}$) the perturbation also enters the region of a nonlinear CL and stabilizes at the level $B_{sat} \sim \gamma_L$.

5. The evolution of perturbations starting from the region of an unsteady CL regime

In this Section we shall consider the fate of a perturbation that starts from the region of an unsteady CL (region II on the diagram), i.e. from values of supercriticalities and initial amplitudes such that

$$\gamma_L > \nu^{\frac{1}{3}}, \quad B \ll \gamma_L^2. \tag{5.1}$$

One should distinguish two cases here: one with no viscosity at all, and the other with some, though small viscosity present. The first case actually means that even a small initial amplitude exceeds the value of $\nu^{\frac{2}{3}}$, i.e. the triple point on the diagram (where all three regimes border each other) can, actually, be transferred to the origin of coordinates. Clearly, the second case is physically more interesting; however, investigating a limiting situation corresponding to the total absence of viscosity is useful in the sense that a substantial part of the evolutionary stage of the perturbation, even in the presence of small viscosity, will be occurring as if it were non-existent, and only for larger times does it become important.

In the inviscid case we expect a limitation of the amplitude growth at a level corresponding to the boundary between the viscous and nonlinear CL, i.e. when $B \sim \gamma_L^2$. Such an expectation is based on a plasma–hydrodynamical analogy between the problem at hand and the problem of the growth of a monochromatic electrostatic wave in a plasma which is driven by an electron beam; this latter problem has been solved previously by two groups of authors: Fried *et al.* (1970) and Onishchenko *et al.* (1970). We have already exploited this analogy in the problem of the nonlinear stability of a zonal flow on a β-plane (Churilov & Shukhman 1987*b*). In the problem of a zonal flow we were successful in establishing a one-to-one correspondence between the parameters of this hydrodynamical problem and the relevant parameters of the above-mentioned plasma problem; more specifically, we were able to demonstrate a total identity of the equation for absolute vorticity with the kinetic equation for the distribution function (for beam electrons). Once such a correspondence had been established, it became possible to extend to the problem of a zonal flow all plasma results obtained by solving numerically the kinetic equation in Onishchenko *et al.* (1970) on a computer. It appears, however, that the problem of a mixing layer in a rotating fluid as treated in the present paper lacks such a simple correspondence with the previously solved plasma problem because in this case one encounters two fundamental differences which do not permit us to make direct use of the plasma results. One is that the problem of our interest involves the displacement effect of the critical layer during the course of the evolution which is

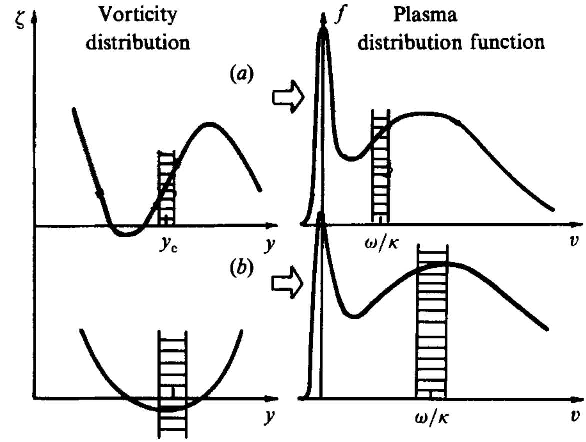

FIGURE 6. Illustration of two different situations of the plasma–hydrodynamic analogy: (*a*) for the case of the β-plane (Churilov & Shukhman 1987*b*), (*b*) for the case considered in the present work.

absent in the plasma problem discussed. The other difference is thus. As we have already pointed out, the plasma–hydrodynamical analogy is based on the similarity of the equations for vorticity with the kinetic equation for the distribution function of particles. In particular, from this analogy it follows that, while in a plasma the instability is associated with the difference from zero of the derivative of the distribution function (the increment is proportional to $\partial f/\partial v$ at the resonance point), in a free shear flow it is proportional to the derivative of vorticity on the CL $\partial \zeta_{00}/\partial y$. But in the plasma problem cited above the position of the resonance region in velocity space corresponds to the middle part of the beam's slope and its width is much less than the distance from maximum $f(v)$. A similar situation occurs on the β-plane (Churilov & Shukhman 1987*b*). (Note that although, owing to small supercriticality, the distance between two zeros of absolute vorticity gradient is small ($O(\gamma_{\mathrm{L}}^{\frac{1}{3}})$) in the β-plane problem considered, the width of the CL is, however, much less than this distance.) In the present case, on the contrary, the position of the critical level is almost coincident with that of an extremum on the profile ζ_{00} (see figures 3 and 6), i.e. its width is the same order of magnitude as, or greater than, the distance between the critical level and the vorticity extremum (whose order is $O(\gamma_{\mathrm{L}})$).† Therefore it becomes impossible here to transfer automatically the plasma results, as in the β-plane case, so that one has to rederive the solution that would apply for the situation presented.

In this region of parameters a scaling that is somewhat different from that made in §3 should be performed. We put

$$\frac{\partial}{\partial t} = -\Omega_{\mathrm{n}}\frac{\partial}{\partial \varphi} + \epsilon^{\frac{1}{2}}\frac{\partial}{\partial \tau}, \quad D = \tfrac{1}{2} + \epsilon^{\frac{1}{2}} D_1, \quad \gamma_{\mathrm{L}} = \epsilon^{\frac{1}{2}}\tilde{\gamma}_{\mathrm{L}}.$$

We write Ψ as $\Psi = \epsilon(\tilde{\Psi}^{(0)} + \epsilon^{\frac{1}{2}}\tilde{\Psi}^{(\frac{1}{2})} + \epsilon\tilde{\Psi}^{(1)} + \ldots)$ and at $O(\epsilon^2)$ of the inner problem obtain

$$\frac{1}{m^2}\frac{\partial \tilde{\zeta}}{\partial \tau} - \frac{\sigma q}{2m}(Y - Y_1)\frac{\partial \tilde{\zeta}}{\partial \theta} + 2C\rho^{-1}\frac{\partial \tilde{\zeta}}{\partial Y} - 2C\sigma D_1 \frac{q}{m}\sin\theta - \frac{4q}{m^2}\frac{\mathrm{d}}{\mathrm{d}\tau}(C\cos\theta) = 0, \quad (5.2)$$

† An analogy with plasma would correspond here to the case where the position of the resonance value of velocity lies slightly to the left of the $f(v)$ maximum (figure 6).

where $\quad \tilde{\Psi}^{(1)\prime\prime} = \tilde{\zeta}+2\left(\frac{2}{m^2}-1\right)C\cos\theta+\frac{\sigma}{2m}q\left(q-\frac{2}{m^2}\right)\rho Y^2, \quad Y_1 \equiv mz_c D_1.$

Using the replacement

$$\tilde{\zeta} = F+4qC\cos\theta-\frac{\sigma q^2}{2m}\left(Y-Y_1-\frac{D_1}{q}\right)^2\rho \tag{5.3}$$

equation (5.2) becomes

$$\frac{\partial F}{\partial \tau}-\tfrac{1}{2}\sigma qm\left(Y-Y_1+\frac{2\omega(\tau)}{q\sigma}\right)\frac{\partial F}{\partial \theta}+2Cm^2\rho^{-1}\sin\theta\frac{\partial F}{\partial Y} = 0. \tag{5.4}$$

The solution of (5.4) must have the asymptotic behaviour

$$F \sim -4qC\cos\theta+\sigma\frac{\rho q^2}{2m}\left(Y-Y_1-\frac{D_1}{q}\right)^2+Y^{-1}\left\{4\left(D_1+\frac{2\omega}{\sigma}\right)C\cos\theta-\frac{8}{m\sigma}\frac{\mathrm{d}C}{\mathrm{d}\tau}\sin\theta\right\}.$$

On supplementing (5.4) with the solvability conditions

$$\int\left\langle\left[F+4qC\cos\theta-\sigma\frac{\rho q^2}{2m}\left(Y-Y_1-\frac{D_1}{q}\right)^2\right]\cos\theta\right\rangle\mathrm{d}Y = -\frac{4m}{q}D_1C+\frac{4}{\mu}\left(\frac{D}{2}+\frac{\omega}{\sigma}\right)C, \tag{5.5}$$

$$\int\left\langle\left[F+4qC\cos\theta-\sigma\frac{\rho q^2}{2m}\left(Y-Y_1-\frac{D_1}{q}\right)^2\right]\sin\theta\right\rangle\mathrm{d}Y = -\frac{4}{\mu m\sigma}\frac{\mathrm{d}C}{\mathrm{d}\tau} \tag{5.6}$$

we obtain the initial system of equations for investigating the evolution of perturbations in the regime with an unsteady CL as well as in the regime that is transitory between an unsteady and nonlinear CL, i.e. in the region of amplitudes $B \sim \gamma_L^2$. It should be borne in mind that the position of the extremum of unperturbed vorticity in these variables corresponds to $Y = Y_1+D_1/q$, and the position of the critical level corresponds to $Y = Y_c(\tau) \equiv Y_1-2\omega(\tau)/\sigma q$. Using the plasma analogy, equation (5.4) resembles the kinetic equation for the case where the wave velocity varies its value during the course of the evolution. This equation no longer contains any small parameters and should be solved numerically.

To ease numerical solution it is convenient to use variables such that the position of the extremum of unperturbed vorticity corresponds to $Y = 0$ and the coordinate of the critical level and the growth rate in the linear stage are unity. It is convenient to measure the amplitude in units proportional to the square of the growth rate. Thus, we put

$$Y = Y_1+\frac{D_1}{q}+\frac{2\tilde{\gamma}_L}{\pi m\mu}Y', \quad \tau = \frac{1}{\tilde{\gamma}_L}\tau', \quad C = \frac{2\rho}{m^3\mu^2\pi^2 q}\tilde{\gamma}_L^2 C',$$

$$F = \frac{4\rho}{m^3\mu^2\pi^2}\tilde{\gamma}_L^2 F', \quad Y_c(\tau') = \frac{2\omega(\tau')/\sigma+D_1}{2\omega_L/\sigma+D_1}, \quad \omega_L \equiv \omega(\tau'=0),$$

and, on dropping the primes, we get

$$\frac{\partial F}{\partial \tau}-\frac{\sigma}{\pi\mu}(Y-Y_c(\tau))\frac{\partial F}{\partial \theta}+\frac{2C}{\pi\mu}\sin\theta\frac{\partial F}{\partial Y} = 0, \tag{5.7}$$

$$\frac{\mathrm{d}C}{\mathrm{d}\tau} = -\frac{1}{\pi}\int\mathrm{d}Y\langle(F+2C\cos\theta-\tfrac{1}{2}\sigma Y^2)\sin\theta\rangle, \tag{5.8}$$

$$Y_c(\tau) = 1+\pi^2\mu^2-\frac{\mu}{C}\int \mathrm{d}Y\langle(F+2C\cos\theta-\tfrac{1}{2}\sigma Y^2)\cos\theta\rangle. \tag{5.9}$$

The system (5.7)–(5.9) must be solved with the initial condition

$$F(\theta, Y;\tau=0) = -2C(0)\cos\theta+\sigma Y^2 + \frac{2C(0)}{\pi^2\mu^2+(Y-1)^2}[-\pi\mu Y\sin\theta+(\pi^2\mu^2+1-Y)\cos\theta]. \tag{5.10}$$

The expression (5.10) (without the term $\sim Y^2$) is an eigenfunction of the linearized problem inside the CL. Indeed, it is easy to verify that when $\tau = 0$, we have $C^{-1}\,\mathrm{d}C/\mathrm{d}\tau = 1, Y_c = 1$.

In order to check the numerical calculation for correctness, it is useful to take account of the integral of the system (5.7)–(5.9) that expresses the law of conservation of energy:

$$W(\tau) \equiv \frac{C^2}{\mu}+\int\left\langle F+2C\cos\theta-\sigma\frac{Y^2}{2}\right\rangle Y\,\mathrm{d}Y = \text{const.} \tag{5.11}$$

Here the first term represents the energy of perturbations in the non-resonance part of the flow which is changed owing to rearrangement of the CL, which is described by the second term of (5.11).

It is convenient to solve (5.7) by using Lagrangian coordinates in a phase space. By considering Y and X (where $X = \theta/2\pi$) to be the coordinates of a particle having at time $\tau = 0$ the coordinates Y_0 and X_0, we find that $F(X, Y;\tau) = F(X_0, Y_0;\tau = 0)$, where the particle trajectories are given by the equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}X(X_0, Y_0;\tau) = -\frac{1}{2\pi^2\mu}[Y(X_0, Y_0;\tau)-Y_c(\tau)], \tag{5.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Y(X_0, Y_0;\tau) = \frac{2C(\tau)}{\pi\mu}\sin[2\pi X(X_0, Y_0;\tau)]. \tag{5.13}$$

Thus, the problem now is to integrate the particle trajectories with the initial conditions $X(X_0, Y_0;0) = X_0$ and $Y(X_0, Y_0;0) = Y_0$ and to substitute, then, into the MSC (5.8) and (5.9):

$$\frac{\mathrm{d}C}{\mathrm{d}\tau} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}Y_0\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{d}X_0\sin(2\pi X)\Big\{E(X, Y;\tau)-E_0(X_0, Y_0) + \frac{2C(0)}{\pi^2\mu^2+(Y_0-1)^2}[-\pi\mu Y_0\sin(2\pi X_0)+(\pi^2\mu^2+1-Y_0)\cos(2\pi X_0)]\Big\}. \tag{5.14}$$

$$Y_c(\tau) = 1+\pi^2\mu^2-\frac{\mu}{C(\tau)}\int_{-\infty}^{\infty}\mathrm{d}Y_0\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{d}X_0\cos(2\pi X)\{\ldots\}. \tag{5.15}$$

Here $\{\ldots\}$ in (5.15) denotes the same bracket as in (5.14), and

$$E(X, Y;\tau) = 2C(\tau)\cos[2\pi X(X_0, Y_0;\tau)]-\tfrac{1}{2}\sigma Y^2(X_0, Y_0;\tau),$$
$$E_0(X_0, Y_0) = 2C(0)\cos(2\pi X_0)-\tfrac{1}{2}\sigma Y_0^2.$$

We integrated the trajectories of 181×21 particles that were originally positioned at nodes of a rectangular grid, with steps along X_0 equal to $\frac{1}{20}$ and with steps along Y_0

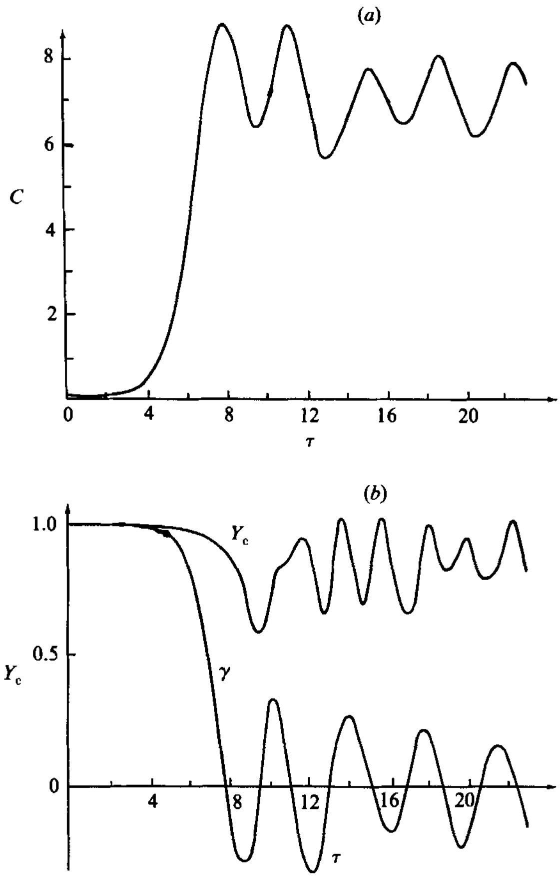

FIGURE 7. The results of a numerical solution of evolution equations (5.7)–(5.9): (a) amplitude $C(\tau)$; (b) increment $\gamma(\tau) = C^{-1}\,\mathrm{d}C/\mathrm{d}\tau$ and the position of the critical level $Y_c(\tau)$.

equal to $2Y_{max}/180$, where $Y_{max} = 10\pi\mu$. The steps in time were chosen equal to 1/4000, and the initial amplitude was assumed to be $C(0) = 0.01$. The results are given in figure 7.

It is evident that the perturbation amplitude C in the initial stage grows exponentially as $\sim \exp(\tau)$; after that, the growth becomes more slow and, upon reaching the level $C_{max} \approx 9$, the amplitude starts to decrease, and this decrease again is replaced by the growth stage, etc. There arise amplitude oscillations around a mean value of $C \approx 7$. The calculations were ceased when $\tau = 23$, when the numerical calculation began to show substantial departures from the law of conservation of energy (5.11). On this time interval the amplitude does not yet show a tendency to assume a stationary value, as is the case for similar calculations reported by Onishchenko *et al.* (1970), but it is also possible that in our case too the amplitude oscillations will ultimately be damped completely and the amplitude will reach its stationary value C_{stat}. Nevertheless, the calculation we have done shows rather confidently that the stability sets in at the level $C_{stat} \approx (6–9)$. In physical variables this saturation amplitude corresponds to

$$B_{sat} = 0.6\gamma_L^2\, C_{sat} \approx (3\text{–}6)\,\gamma_L^2. \qquad (5.16)$$

A physical mechanism for this stabilization is the phase mixing of liquid particles, each of which, in the absence of viscosity, carries its own vorticity. This mixing is produced owing to asynchronism of the motion of the particles along different streamlines. As a result, the vorticity profile inside the CL becomes severely jagged, which decreases the effective mean value of ζ'. In plasma physics this mechanism for phase mixing is a well-known one, see e.g. Galeev & Sagdeev (1973), and Kadomtsev (1976). (For hydromechanics the effect of the decreasing scale over which the vorticity varies was demonstrated analytically by Stewartson 1978.)

The picture outlines above is produced in the case where viscosity is totally neglected. Now, we shall take it into account. (Regrettably, the following treatment has a qualitative character, i.e. is without numerical calculations.) Viscosity gives rise to two effects. First, it removes the jaggedness on the vorticity profile. Second, it tends to impart to the vorticity profile inside the CL the same slope as in its immediate neighbourhood. The sign of this slope, on the average, is such that it coincides with the sign of the slope of the unperturbed vorticity that has caused the instability and, therefore, the perturbation will continue to grow, though slowly. As a result, the CL will become a nonlinear† one and will, hence, be described by the equations derived in §4.2 for the nonlinear CL, i.e. equations (4.5) and (4.6). In the end, the amplitude will reach the level $B \sim \gamma_L$ and the growth will stop. Thus, the main difference of the evolution of a perturbation that starts from the region of an unsteady CL ($\gamma_L > \nu^{\frac{1}{3}}$) from the evolution of a perturbation that starts from the region of a viscous CL ($\gamma_L < \nu^{\frac{1}{3}}$) lies in the fact that, in the second case, the amplitude grows monotonically, while in the first case it is accompanied by oscillations. In either case, however, the perturbation is stabilized at the level $B_{sat} \sim \gamma_L$ when $\gamma_L > \nu^{\frac{2}{3}}$, or at the level $B_{sat} \sim (\gamma_L \nu^2)^{\frac{1}{4}}$ when $\gamma_L < \nu^{\frac{2}{3}}$.

6. Discussion

Thus, we have shown that, in a weakly supercritical flow of a rotating fluid, equilibrium is reached at a low level that is proportional to a certain positive power of the instability growth rate γ_L.

Let us discuss the physical meaning of the stabilization mechanism leading to the saturation level $B_{sat} \sim \gamma_L$. This can be done with the aid of figure 8, showing on an enlarged scale the unperturbed vorticity profile near the minimum (to be more specific, we assume $\sigma > 0$). In the vicinity of this minimum lies the initial position of the critical level. For convenience, in figure 8 this minimum is placed at the origin of coordinates along the y-axis. The position of the critical level at the initial moment of time corresponds to $y_c = -(2(\Delta\Omega_p)_L + \Delta D)/q > 0$. The instability in the linear stage (i.e. when the scale of the CL still is much smaller than its distance y_c from the position of the unperturbed vorticity minimum) is due to the positive sign of the slope of the profile on the CL. What occurs later as the amplitude is growing?

From the calculations performed in §4 it follows that, as the amplitude reaches values corresponding to the nonlinear CL, the profile of mean vorticity is deformed in such a way that its minimum becomes coincident with the current position of the critical level. In this case the growth rate diminishes, $\gamma \rightarrow \gamma_L(\nu/B^{\frac{3}{2}})$; however, it nevertheless remains different from zero because viscous forces tend to impart to the

† More explicitly, this means that the nonlinear term will become the leading one (rather than of comparable importance to unsteady terms as before) and even viscous terms will be greater than unsteady terms, i.e. the hierarchy of terms will become such as in §4.2.

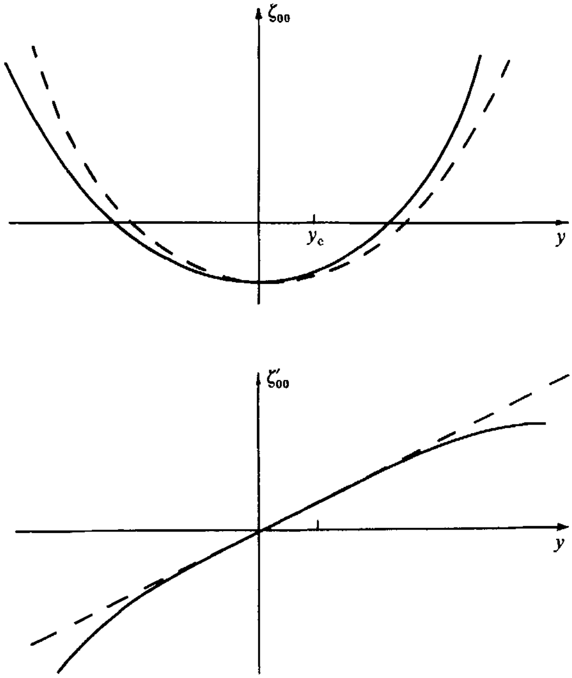

FIGURE 8. Profiles of unperturbed vorticity and its derivative in the neighbourhood of the CL (the case $\sigma > 0$). Dashes correspond to a purely quadratic approximation of the profile near the minimum.

vorticity profile a slope which is dictated by the slope to the right and to the left of the CL. On the average (with respect to the CL width), this slope is positive because the critical level is shifted to the right of the position of the unperturbed profile minimum, and this leads to the fact that the amplitude growth, though a power law, continues. This stage is described by (4.5) and (4.6) without the contribution with L_3. If the unperturbed profile ζ_{00} were a 'pure' parabola, i.e. without the addition of cubic and other terms, then the perturbation would continue to grow according to this law to reach amplitudes of the order unity, i.e. to the limit where the validity range of weakly nonlinear theory no longer holds. Next, we take into consideration that the expansion of ζ_{00} near the minimum involves cubic terms. This means that the profile ζ'_{00} deviates slightly from a straight line (with a positive slope). Let us further imagine that the separatrix is already so wide that the negative slope to the left of the 'cat's eye' is able to compensate the positive slope on the right. Now, we determine what the sign of ζ'''_{00} must be and what value of the amplitude is needed for this.

Let the unperturbed vorticity profile near the minimum be of the form

$$\zeta_{00} = \text{const} + \zeta'_{00}\, s + \tfrac{1}{2}\zeta''_{00}\, s^2 + \tfrac{1}{6}\zeta'''_{00}\, s^3 \quad (\zeta'_{00} > 0, \zeta''_{00} > 0) \tag{6.1}$$

(the derivatives are taken at the point $y = y_c, s = y - y_c$). We again recall that $\zeta'_{00} \sim \gamma_L$. Let L be the semi-width of the separatrix ($L \sim O(B^{\frac{1}{2}})$). The balance condition for the slopes, then, means

$$\zeta'_{00}(s_c - L) + \zeta'_{00}(s_c + L) = 0, \tag{6.2}$$

where s_c is a new position of the critical level. Taking into consideration that $s_c \sim O(\gamma_L)$, from (6.2) we get

$$\gamma_L \sim -\zeta'''_{00} L^2. \tag{6.3}$$

From (6.3) it follows that stabilization is possible whenever $\zeta'''_{00} < 0$. In this case the width of the separatrix is $L \sim O(\gamma_L^{\frac{1}{2}})$, which corresponds to the amplitude $B_{sat} \sim O(\gamma_L)$. The meaning of the condition $\zeta'''_{00} < 0$ is clear from figure 8. Indeed, stabilization requires that taking into account the cubic terms in the vorticity expansion (or quadratic terms in ζ'_{00}) increases, in absolute value, the negative slope to the left of the CL and decreases the positive slope to the right, i.e. the profile ζ'_{00} must be *convex upwards*: $(\zeta'_{00})'' < 0$. Taking into account the second variant when $\sigma < 0$ (i.e. when $\zeta'_{00} < 0$ at the critical level), the criterion for the instability being stabilized at a low level may be written as: $(\zeta'''_{00}\zeta'_{00})_c < 0$. The considerations presented here have a rather crude character; they would be totally valid for plane geometry. For circular geometry, as an exact calculation demonstrates (see (3.22) and (3.16)), this criterion is modified to become

$$[(\Delta_y \zeta_{00})' \zeta'_{00}]_c < 0. \tag{6.4}$$

It is this relationship of the signs that occurs in the model we have considered here.

From the foregoing discussion it follows that the results obtained in this paper are not tied to the particular model but have largely a universal character. Only the coefficients in the linear relationships relating the instability growth rate and the phase velocity to the supercriticality parameter depend on the chosen model as well as the explicit form of the derivatives in the expansion of the vorticity profile on the CL of the form (6.1). These latter, provided that the criterion (6.4) is satisfied, determine the numerical value of the coefficient α in the relationship $B_{sat} = \alpha\gamma_L$. It is understandable that the reason for such universality is provided by the fact that perturbation dynamics is determined mainly by the flow rearrangement inside of the CL and does not depend on the flow structure as a whole. Such a property is inherent in all free flows with critical layers.

In summarizing the foregoing treatment, we wish to make a remark concerning the region of very small increments, i.e. the part of figure 4 in which $\gamma_L < \nu^{\frac{4}{3}}$. In this region the amplitude equation (4.1) is inapplicable because it involves a stronger nonlinearity leading to the equation

$$\frac{dB}{dt} = \gamma_L B - \frac{a}{\nu^{\frac{1}{3}}} B^3.$$

The origin of this nonlinearity was discussed in papers by Churilov & Shukhman (1987*c*) and Huerre (1987). As a result, in the region $\gamma_L < \nu^{\frac{4}{3}}$ the saturation level is found to be $B^{(3)}_{sat} \sim (\gamma_L \nu^{\frac{1}{3}})^{\frac{1}{2}}$.

I thank Dr I. I. Pasha, together with whom the flow model considered here was found, and Dr. S. M. Churilov for numerous fruitful discussions. Special thanks are due to Mr V. G. Mikhalkovsky for his assistance in preparing the English version of the manuscript and for typing and retyping the text.

Appendix. Solving the equation for the inner region for the viscous and nonlinear CL regimes and calculating the integrals $\int\langle\tilde{\zeta}\sin\theta\rangle\,\mathrm{d}Y$ and $\int\langle\tilde{\zeta}\cos\theta\rangle\,\mathrm{d}Y$.

The right-hand side of (3.9) involves three terms which are responsible for the origin of the functions Φ_1, Φ_2, and Φ_3:

$$-\frac{\sigma}{2m}q\rho Y\frac{\partial\tilde{\zeta}}{\partial\theta}+2C\sin\theta\frac{\partial\tilde{\zeta}}{\partial Y}-\eta\frac{\partial^2\tilde{\zeta}}{\partial Y^2}=R_1+R_2+R_3, \tag{A 1}$$

$$R_1=\frac{4q\rho}{m}\left(\omega+\frac{D_1\sigma}{2}\right)C\sin\theta,\quad R_2=\frac{4q\rho}{m^2}\frac{\mathrm{d}C}{\mathrm{d}\tau}\cos\theta,\quad R_3=-\frac{16q}{m}C^2\sin 2\theta.$$

Since (A 1) is linear with respect to $\tilde{\zeta}$, then $\tilde{\zeta}$ can be divided into three contributions $\tilde{\zeta}_1$, $\tilde{\zeta}_2$, and $\tilde{\zeta}_3$, each of which obeys the equation with a relevant right-hand side. Let us consider three contributions separately.

A.1. *The contribution of $\tilde{\zeta}_1$, the function $\Phi_1(x)$*

Through the replacement

$$\tilde{\zeta}_1=\zeta_1+\frac{2q\rho}{m}(\omega+\tfrac{1}{2}D_1\sigma)\,Y$$

the equation for ζ_1 becomes

$$-\frac{\sigma q\rho}{2m}Y\frac{\partial\zeta_1}{\partial\theta}+2C\sin\theta\frac{\partial\zeta_1}{\partial Y}-\eta\frac{\partial^2\zeta_1}{\partial Y^2}=0 \tag{A 2}$$

and ζ_1 must have an asymptotic representation

$$\zeta_1\sim-\frac{2q\rho}{m}(\omega+\tfrac{1}{2}D_1\sigma)\,Y+O(Y^{-1}). \tag{A 3}$$

After an appropriate change of the notation, the problem (A 2) and (A 3) becomes exactly the one solved by Haberman (1972). Details can be found in the paper just cited; we give here the result:

$$\left.\begin{aligned}\int\langle\tilde{\zeta}_1\sin\theta\rangle\,\mathrm{d}Y&=-4C\sigma\left(\frac{D_1}{2}+\frac{\omega}{\sigma}\right)\Phi_1\left(\frac{C}{(p\eta^2)^{\frac{1}{3}}}\right),\\ \int\langle\tilde{\zeta}_1\cos\theta\rangle\,\mathrm{d}Y&=0,\end{aligned}\right\} \tag{A 4}$$

where $\Phi_1(x)=\Phi_{\mathrm{H}}(x^{-\frac{3}{2}})$, and $\Phi_{\mathrm{H}}(\lambda)$ is the Haberman function.

A.2. *The contribution of $\tilde{\zeta}_2$, the function $\Phi_2(x)$*

For $\tilde{\zeta}_2$ we have

$$-\frac{\sigma}{2m}q\rho Y\frac{\partial\tilde{\zeta}_2}{\partial\theta}+2C\sin\theta\frac{\partial\tilde{\zeta}_2}{\partial Y}-\eta\frac{\partial^2\tilde{\zeta}_2}{\partial Y^2}=\frac{4q\rho}{m^2}\gamma(\tau)\,C\cos\theta. \tag{A 5}$$

A.2.1. *The limit of a nonlinear CL ($C/\eta^{\frac{2}{3}}\gg 1$)*

For definiteness, assume $\sigma>0$ (the final result (A 20) holds true for both signs of σ). We denote $p=q\rho/2m$, $Z=p^{\frac{1}{2}}C^{-\frac{1}{2}}Y$, and $\tilde{\zeta}_2=C^{\frac{1}{2}}G$. For G we have

$$2\sin\theta\frac{\partial G}{\partial Z}-Z\frac{\partial G}{\partial\theta}=p^{\frac{1}{2}}\left(\frac{\eta}{C^{\frac{3}{2}}}\frac{\partial^2 G}{\partial Z^2}+\frac{8}{m}\frac{\gamma}{C}\cos\theta\right). \tag{A 6}$$

In the limit of a nonlinear CL the quantities $\eta/C^{\frac{3}{2}}$ and γ/C are small parameters, and the smallness is of the same order of magnitude. We put

$$\kappa = Z^2 - 4\cos\theta, \quad Z = \pm(\kappa + 4\cos\theta)^{\frac{1}{2}}. \tag{A 7}$$

In the variables θ and κ, (A 6) has the form

$$-\left(\frac{\partial G}{\partial \theta}\right)_\kappa = 4p^{\frac{1}{2}}\left\{\frac{2}{m}\frac{\gamma}{C}\frac{\cos\theta}{Z(\theta,\kappa)} + \frac{\eta}{C^{\frac{3}{2}}}\frac{\partial}{\partial\kappa}Z(\theta,\kappa)\frac{\partial G}{\partial \kappa}\right\}. \tag{A 8}$$

Then, we proceed according to perturbation theory

$$G = G^{(0)} + G^{(1)} + \dots .$$

We have
$$G^{(0)} = G^{(0)}(\kappa), \tag{A 9}$$

where $G^{(0)}(\kappa)$ still is an arbitrary function κ, which we determine from the solvability condition of the equation for $G^{(1)}$.

Outside the separatrix ($\kappa > 4$) we obtain (indices e and i will henceforth denote the quantities taken outside and inside the separatrix, respectively):

$$\frac{\eta}{C^{\frac{3}{2}}}\frac{\partial}{\partial\kappa}Q_e(\kappa)\frac{\partial G_e^{(0)}}{\partial\kappa} + \frac{2}{m}\frac{\gamma}{C}P_e(\kappa) = 0, \tag{A 10}$$

where
$$Q_e(\kappa) = \int_{-\pi}^{\pi} Z(\theta,\kappa)\,d\theta = \pm\tilde{Q}_e(\kappa), \tag{A 11}$$

$$P_e(\kappa) = \int_{-\pi}^{\pi}\frac{\cos\theta}{Z(\theta,\kappa)}d\theta = \pm\tilde{P}_e(\kappa). \tag{A 12}$$

For $\kappa \gg 1$, we have $\tilde{Q}_e(\kappa) = 2\pi\kappa^{\frac{1}{2}}(1+O(\kappa^{-2}))$ and $\tilde{P}_e = -2\pi\kappa^{-\frac{3}{2}}$. From (A 10) we get

$$G_e^{(0)}(\kappa) = \frac{2\gamma}{m}\frac{C^{\frac{1}{2}}}{\eta}\int_4^\kappa \frac{ds}{Q_e(s)}\int_s^\infty P_e(s')\,ds' + G_0, \tag{A 13}$$

where $G_0(\tau)$ is the 'constant' of integration. Note that, when $\kappa \gg 1$, the asymptotic $G^{(0)}$ has the form

$$G^{(0)} \sim -\frac{4\gamma}{m}\frac{C^{\frac{1}{2}}}{\eta}\ln\kappa \tag{A 14}$$

or, in the initial variables,
$$\tilde{\zeta} = -\frac{4}{m}A(Y)\frac{1}{\eta}\frac{dC^2}{d^2\tau}. \tag{A 15}$$

No such contribution is present in the asymptotic dictated by the solution of the outer problem. It appears that in order to achieve an accurate matching of the zero harmonic, it is necessary to introduce an intermediate region, whose position l is determined from the balance condition for viscous and time-derivative terms, i.e. $\partial/\partial\tau \sim \nu/l^2$, whence we have $l \sim (\nu/\gamma)^{\frac{1}{2}} \sim (\epsilon^{\frac{3}{2}}/\epsilon)^{\frac{1}{2}} \sim \epsilon^{\frac{1}{4}}$ (it must be recalled that the CL width is $O(\epsilon^{\frac{1}{2}})$). We have already faced the need to introduce the intermediate region for a correct matching of the zero harmonic in previous papers (Churilov & Shukhman 1987*a*, *b*). However, it is no longer necessary to do these calculations in explicit form in the following.

Next, inside the separatrix ($-4 < \kappa < 4$) we have

$$G_i^{(0)} = \frac{2\gamma}{m}\frac{C^{\frac{1}{2}}}{\eta}\int_\kappa^4 \frac{ds}{Q_i(s)}\int_{-4}^s P_i(s')\,ds' + G_0, \tag{A 16}$$

where $$Q_i(\kappa) = \frac{1}{2}\oint Z(\theta,\kappa)\,\mathrm{d}\theta, \quad P_i(\kappa) = \frac{1}{2}\oint \frac{\cos\theta}{Z(\theta,\kappa)}\,\mathrm{d}\theta, \tag{A 17}$$

and the constant of integration G_0 is the same as in (A 13). This follows from the requirement of continuity of G on the 'cat's eye' boundary, $\kappa = 4$. As a result, we obtain
$$\oint\langle\tilde{\zeta}_2\cos\theta\rangle\,\mathrm{d}Y = 4C\frac{\gamma}{m}I_2\left[\frac{C}{(p\eta^2)^{\frac{1}{3}}}\right]^{\frac{3}{2}}, \quad \oint\langle\tilde{\zeta}_2\sin\theta\rangle\,\mathrm{d}Y = 0, \tag{A 18}$$

where $I_2 = I_e + I_i$,

$$I_e = \frac{1}{2\pi}\int_4^\infty \frac{\mathrm{d}s}{\tilde{Q}_e(s)}\left\{\int_s^\infty \tilde{P}_e(t)\,\mathrm{d}t\right\}^2$$
$$= \frac{16^2\sqrt{2}}{\pi}\int_0^1 \frac{\mathrm{d}k}{k^2\mathbb{E}}\left\{\int_0^k \frac{\mathrm{d}q}{q^2}\left[\left(1-\frac{2}{q^2}\right)\mathbb{K}(q)+\frac{2}{q^2}\mathbb{E}(q)\right]\right\}^2 = 4.26,$$

$$I_i = \frac{1}{2\pi}\int_{-4}^4 \frac{\mathrm{d}s}{Q_i(s)}\left\{\int_{-4}^4 P_i(t)\,\mathrm{d}t\right\}^2$$
$$= \frac{16^2\sqrt{2}}{\pi}\int_0^1 \frac{k\,\mathrm{d}k}{\mathbb{E}(k)-(1-k^2)\mathbb{K}(k)}\left\{\int_0^k q\,\mathrm{d}q[2\mathbb{E}(q)-\mathbb{K}(q)]\right\}^2 = 9.84$$

and $\mathbb{K}(x)$ and $\mathbb{E}(x)$ are complete elliptic integrals.

A.2.2. *The limit of a viscous CL* ($C/\eta^{\frac{2}{3}} \ll 1$)

Here it is convenient to use the variable $x = kY$, $k = -(p/\eta)^{\frac{1}{3}} = -(q\rho/2m\eta)^{\frac{1}{3}}$. As a result, (A 5) assumes the form
$$\frac{\partial^2\tilde{\zeta}_2}{\partial x^2} - x\frac{\partial\tilde{\zeta}_2}{\partial\theta} = \frac{2C}{\eta k}\sin\theta\frac{\partial\tilde{\zeta}_2}{\partial x} + 8\frac{Ck}{m}\gamma\cos\theta.$$
Calculations with a 'viscous' operator appearing on the left-hand side of this equation are more traditional and have already been given in a number of papers (Haberman 1976; Churilov & Shukhman 1987*c*). The main nonlinear contribution here is due to the zeroth harmonic, i.e. to the distortion of a mean flow. Dropping the details, we give the result to an accuracy of terms cubic in amplitude:
$$\left.\begin{aligned}\oint\langle\tilde{\zeta}_2\cos\theta\rangle\,\mathrm{d}Y &= 4C\frac{\gamma(\tau)}{m}\left(\pi + I_0\frac{C^2}{(p\eta^2)^{\frac{2}{3}}}\right),\\ \oint\langle\zeta_2\sin\theta\rangle\,\mathrm{d}Y &= 0,\end{aligned}\right\} \tag{A 19}$$

where $$I_0 = \int_{-\infty}^{\infty} |F(x)|^2\,\mathrm{d}x = \pi(\tfrac{2}{3})^{\frac{2}{3}}\Gamma(\tfrac{1}{3}) = 6.42$$

and $F(x)$ is a solution of the equation $F'' - \mathrm{i}xF = -\mathrm{i}$ with the asymptotic $F = 1/x$ when $x \to \pm\infty$. By joining (A 18) with (A 19), we finally obtain:

$$\left.\begin{aligned}\oint\langle\tilde{\zeta}_2\cos\theta\rangle\,\mathrm{d}Y &= -4C\frac{\gamma}{m}\Phi_2(x), \quad x = C/(p\eta^2)^{\frac{1}{3}},\\ \Phi_2(x) &= \begin{cases}-(\pi + I_0x^2), & x \ll 1\\ -I_2x^{\frac{3}{2}}, & x \gg 1.\end{cases}\end{aligned}\right\} \tag{A 20}$$

A.3. *The contribution of* $\tilde{\zeta}_3$, *the function* $\Phi_3(x)$

Through the replacement

$$\tilde{\zeta}_3 = \zeta_3 + \frac{4}{3}\frac{\sigma}{m^2}q^2\rho Y^3 - \frac{16q}{m}CY\cos\theta$$

the equation for the function $\tilde{\zeta}_3$ becomes

$$-\frac{\sigma}{2m}q\rho Y\frac{\partial\tilde{\zeta}_3}{\partial\theta} + 2C\sin\theta\frac{\partial\tilde{\zeta}_3}{\partial Y} = \eta\left(\frac{\partial^2\tilde{\zeta}_3}{\partial Y^2} + \frac{8\sigma}{m^2}q^2\rho Y\right), \tag{A 21}$$

with the asymptotic

$$\tilde{\zeta}_3 \sim -\frac{4}{3}\frac{\sigma}{m^2}q^2\rho Y^3 + \frac{16q}{m}CY\cos\theta. \tag{A 22}$$

A.3.1. *The limit of a nonlinear CL*

Through replacements similar to those when calculating $\tilde{\zeta}_2$, (A 21) is reduced to the equation (for definiteness, we again assume $\sigma > 0$)

$$-\left(\frac{\partial G}{\partial\theta}\right)_\kappa = \frac{4\eta}{C^{\frac{3}{2}}}\left\{p^{\frac{1}{2}}\frac{\partial}{\partial\kappa}Z(\theta,\kappa)\frac{\partial G}{\partial\kappa} + \frac{4q}{m}\right\}, \tag{A 23}$$

with the asymptotic G when $\kappa \gg 1$

$$G \sim \mp\frac{8}{3}\frac{q}{mp^{\frac{1}{2}}}\kappa^{\frac{3}{2}}. \tag{A 24}$$

We have

$$G_e^{(0)} = -\frac{8\pi q}{mp^{\frac{1}{2}}}\int_4^\kappa\frac{s\,ds}{Q_e(s)} + G_0, \tag{A 25}$$

$$G_i^{(0)} = G_0. \tag{A 26}$$

The solution $G_i^{(0)}$ is obtained from the condition that no singularity is present at the centre of the 'cat's eye'. It is convenient to separate from (A 25) the asymptotic (A 24) in explicit form:

$$G_e^{(0)} = \mp\frac{8}{3}\frac{q}{mp^{\frac{1}{2}}}\kappa^{\frac{3}{2}} \mp \frac{8\pi q}{mp^{\frac{1}{2}}}\left\{\frac{8}{3\pi} + \int_4^\kappa\left[\frac{1}{Q_e(s)} - \frac{1}{2\pi s^{\frac{1}{2}}}\right]s\,ds\right\} \tag{A 27}$$

so that when $\kappa \gg 1$

$$G_e^{(0)} = \mp\frac{8}{3}\frac{q}{mp^{\frac{1}{2}}}\kappa^{\frac{3}{2}} + R^{\pm}, \tag{A 28}$$

where the jump $R^+ - R^-$ is†

$$R^+ - R^- = \frac{8q}{mp^{\frac{1}{2}}}I_3, \quad I_3 = 2\pi\left\{\frac{8}{3\pi} + \int_4^\infty s\,ds\left(\frac{1}{2\pi s^{\frac{1}{2}}} - \frac{1}{\tilde{Q}_e(s)}\right)\right\} = 4.26.$$

It is easy to establish the relation of the jump $R^+ - R^-$ to the desired integral $\int\langle\zeta_3\sin\theta\rangle\,dY$. As in Haberman's (1972) paper it is given by the relationship

$$\int\langle G\sin\theta\rangle\,dZ = \frac{\eta}{2C^{\frac{3}{2}}}p^{\frac{1}{2}}(R^+ - R^-). \tag{A 29}$$

† The asymptotic as dictated by the outer solution, i.e. (A 24) involves no contributions of the form $R^{\pm}$. A correct matching here again requires an intermediate region.

Finally, we obtain (for both signs of σ)

$$\int \langle \tilde{\zeta}_3 \sin\theta \rangle \, \mathrm{d}Y = \frac{8C^2}{\sigma\rho}\left[\frac{C}{(p\eta^2)^{\frac{1}{3}}}\right]^{-\frac{3}{2}} I_3, \tag{A 30}$$

$$\int \langle \zeta_3 \cos\theta \rangle \, \mathrm{d}Y = 0. \tag{A 31}$$

A.3.2. *The limit of a viscous CL*

Dropping the calculation, we shall give the result only. Here the main nonlinear contribution is also due to the distortion of the mean flow and is fifth order in amplitude:

$$\int \langle \tilde{\zeta}_3 \sin\theta \rangle \, \mathrm{d}Y = \frac{8C^2}{\sigma\rho}\left[\frac{C}{(p\eta^2)^{\frac{1}{3}}}\right]^{3} I_4, \quad \int \langle \zeta_3 \cos\theta \rangle \, \mathrm{d}Y = 0, \tag{A 32}$$

where
$$I_4 = -4\int_{-\infty}^{\infty} (\operatorname{Im} F)(\operatorname{Im} H)\, \mathrm{d}x = 1.12\pi,$$

and the functions $F(x)$ and $H(x)$ satisfy the equations

$$F'' - \mathrm{i}xF = -\mathrm{i}, \quad H'' - \mathrm{i}xH = \mathrm{i}\, 2^{-\frac{2}{3}} \frac{\mathrm{d}}{\mathrm{d}x} F(2^{\frac{1}{3}}x).$$

Upon combining (A 30) and (A 32), we get

$$\left.\begin{aligned} \int \langle \tilde{\zeta}_3 \sin\theta \rangle \, \mathrm{d}Y &= \frac{8C^2}{\sigma\rho} \Phi_3(x), \quad x = C/(p\eta^2)^{\frac{1}{3}}, \\ \Phi_3(x) &= \begin{cases} I_4 x^3, & x \ll 1, \\ I_3 x^{-\frac{3}{2}}, & x \gg 1. \end{cases} \end{aligned}\right\} \tag{A 33}$$

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